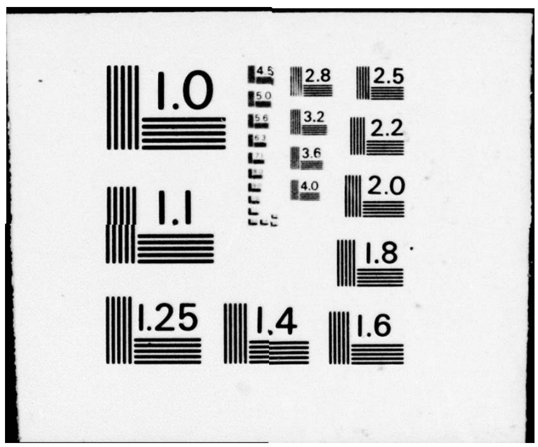
AD-A033 392 CORNELL UNIV ITHACA N Y DEPT OF OPERATIONS RESEARCH F/G 5/4
AN ANALYSIS OF WEIGHTED VOTING SYSTEMS USING THE BANZHAF VALUE.(U)
SEP 76 E A WALTHER N00014-67-A-0077-0014
UNCLASSIFIED TR-309 NL

1 OF 1
ADA033392

END
DATE
FILMED
2 - 77


1.0
1.1
1.25
1.4
1.6
1.8
2.0
2.2
2.5
2.8
3.2
3.6
4.0
4.5
5.0
5.6

ADA033392

DEPARTMENT OF OPERATIONS RESEARCH

COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853

DISTRIBUTION STATEMENT A
Approved for public release;
Distribution Unlimited

SCHOOL OF OPERATIONS RESEARCH
AND INDUSTRIAL ENGINEERING
COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
ITHACA, NEW YORK

TECHNICAL REPORT NO. 309

September 1976

AN ANALYSIS OF WEIGHTED VOTING SYSTEMS USING THE BANZHAF VALUE

by

Eleanor Ann Walther

Sec 1473

Research supported in part by the National Science Foundation under grant MPS75-02024 and the Office of Naval Research under contract N00014-75-C-0678.

ABSTRACT

In 1962, the Supreme Court in its decision of Baker versus Carr opened the door for reapportionment by population. In 1967, the New York Courts ruled that weighted voting systems could be implemented in reapportionment schemes. The New York Courts have also accepted the Banzhaf value as a measure of voting power.

This thesis defines the Banzhaf value and discusses some of its properties. An analysis of existing weighted voting schemes for four New York Counties are presented. New schemes are proposed. The appendix presents the method used to compute the Banzhaf value. The computer programs are also included in the appendix.

TABLE OF CONTENTS

LIST OF TABLES

INTRODUCTION

Traditional political philosophy has two tasks: (1) determination of what justice is and (2) determination of whether particular instruments are appropriate for realizing given concepts of justice. Unfortunately, political scientists do not agree to what is justice. Thus quantifying the notions associated with justice is no easy task. In particular, many reasonable definitions of power have been proposed.

In his article, "Some Ambiguities of Power" [13], Riker presents five different measures of voting power. Three of these concern themselves with one's ability to decrease another's utility or to force another to do something he/she would not ordinarily do. The other two concern themselves with the ability to influence or control a particular outcome. Riker shows that with the five definitions, there are at least four distinct meanings, each of which appears quite reasonable by itself. Riker shows that the five definitions presented in his article are not consistent. Thus he is pessimistic about finding a more abstract or general definition of power that would be consistent with each of the proposed definitions of power. If we cannot agree on the definition of power, then how can we ever hope to design a voting system where each person has equal "power"?

The following are reasonable criteria which might be embodied in a voting scheme, in particular, a weighted voting scheme: (1) the financial stake of the participants, (2) the effect the decision will have on the participants, (3) the expertise of the various members on a particular issue, (4) the seniority of the members, (5) the personal power of the participants (One may wish to formalize this rather than have it exercised informally by influencing the votes of others.), (6) the ability of a

participant to carry out the decisions, and (7) the size of the organization a participant represents.

The Supreme Court of the United States has mandated that the equal protection clause of the federal constitution applies to every citizen at all levels of government. Thus they have given us a guideline for formulating a definition of power.

The unprecedented decision of the United States Supreme Court in Baker v. Carr [1] opened the door for reapportionment at all levels of government. The "one person, one vote" phrase which has become the jargon for reapportionment decisions was first used in a Georgia voting case decided a year after the Baker case: "The conception of political equality from the Declaration of Independence to Lincoln's Gettysburg Address, to the Fifteenth, Seventeenth, and Nineteenth Amendments can only mean one thing--one person, one vote." (Gray v. Sanders) [6].

The Baker v. Carr case challenged a 1901 Tennessee statut apportioning the Tennessee General Assembly. The contention was that inequalities in the populations of assembly districts diluted votes of constituents in larger districts, thereby violating the equal protection clause of the United States Constitution. A three-judge federal court dismissed the complaint. On appeal the Supreme Court ruled that the district court had jurisdiction over the subject matter, that the plaintiffs, as qualified voters in the allegedly diluted districts, had standing to bring suit, and that the alleged denial of equal protection presented a "justiciable constitutional cause of action."

Since that time there has been a flood of litigation concerning reapportionment cases in federal courts. The bulk of these cases challenged the structure of bicameral legislatures in which the means of apportioning

legislatures were not based on population. The federal courts have ruled that the "one person, one vote" concept applies to local governments, school district trustees, and political party structures.

In 1967, the Court of Appeals of New York ruled (Graham v. Board of Supervisors of Erie County [5]) that weighted voting may be approved "solely as a temporary expedient; but that a permanent plan must be based on the principle of 'one man, one vote'". At that time weighted voting could only be employed as a stopgap measure, to be used while a new apportionment system was being instituted. A year later, in a landmark case (Town of Greenburgh v. Board of Supervisors) [17] the New York Court of Appeals upheld the constitutionality of a weighted voting system based on population, (This is the only form of weighted voting that the New York courts have approved.) At present at least twenty-two of the fifty-seven counties of New York (outside of New York City) have adopted weighted voting schemes for their county board of legislators. However to conform with the ruling of the Court of Appeals ruling in Iannucci v. Board of Supervisors (1967) [7], a computerized analysis must be presented to validify the Apportionment plan.

> "It is improper for a court in passing upon a constitutional question, to lightly disregard the considered judgement of a legislative body which is also charged with duty to uphold the Constitution but with respect to weighted voting a considered judgement is impossible without computer analysis and, accordingly if county board of supervisors chose to reapportion themselves by use of weighted voting there is no alternative but to require them to come forward with such analysis and demonstrate the validity of their apportionment plan."

SOME PRELIMINARIES

We will now define some terms and describe several notions associated with weighted voting systems.

Defintions

A weighted majority game is denoted by

$$[q;\ w_1, w_2, \ldots, w_n]$$

Here there are n players (voters, or districts). The weight given district i is w_i. The quota of the game is denoted by q. It is generally assumed that

$$q > \frac{\sum_{i=1}^{n} w_i}{2} \quad {}^{1}$$

and that $q > w_i$ for all i.

A subset of n voters may form a bloc called a coalition. A coalition, S, has weight equal to

$$\sum_{i \varepsilon S} w_i$$

A coalition is called winning if

---

1 $\sum_{i=1}^{n} w_i$ means the sum of the terms $w_1, w_2, \ldots, w_n$. That is

$$\sum_{i=1}^{n} w_i = w_1 + w_2 + \ldots + w_n.$$

$$\sum_{i \varepsilon S} w_i \geq q.$$

Note that if we require

$$q > \frac{\sum_{i=1}^{n} w_i}{2}$$

then there is at most one winning coalition in a partition of voters. A coalition is <u>minimal winning</u> if a coalition is winning and no proper subset of the coalition is winning. A coalition, S, is <u>losing</u> if $\sum_{i \varepsilon S} w < q$. A coalition, S, has <u>veto power</u> if N-S is not winning, where N stands for the coalition of all players. (A veto power coalition need not be winning.) A player, j, is a <u>dictator</u> if $w_j > q$. A player is a <u>dummy</u> if he/she is in no winning coalitions.

<u>Notion of Voting Power versus Weight</u>

Weighted voting has been used to correct the disparity between districts which have unequal populations for which the "one person, one vote" concept fails. Unfortunately, in the past, systems have been set up which equate voting power and the number of votes a district receives. The two need not be equivalent. Consider the following games:

[2; 1, 1, 1]

In this game each player is clearly symmetrical. Each player's weight is 1/3 of the total weight. And his/her voting power is clearly equal. But consider

[3; 2, 2, 1].

In this game player 1 and player 2 each have 2/5 of the total weight. Any two players can form a winning coalition. Therefore no one player is more powerful than another. Thus each player's voting power can again be thought of as 1/3. (We assume the sum of all the players' power equals 1.) We can exaggerate this point still further with the following game.

$$[100;\ 99,\ 98,\ 2]$$

The same phenomena occurs. Two players are needed to form a winning coalition. So once again each player's voting power is 1/3.

Thus we can see it is fallacious to assume that voting power and the number of votes a player casts are synonymous. Since there is not a direct proportionality between voting power and weight, one must define a measure of voting power. Each person's voting power should be equal. A player's voting power is associated with his/her ability to be critical in a voting situation. By critical we mean that a player changes the outcome of the vote on an issue by changing his/her vote.

THE BANZHAF VALUE

The Banzhaf index (value, number) was introduced by John Banzhaf in an article appearing in the Rutgers Law Review [2] in 1965. He used the Banzhaf index to argue that weighted voting systems have been misused.

The Banzhaf index is concerned with the 2^n combinations of yes or no votes possible in a game with n players. Each player may either vote for or against an issue. A player is called a swinger if by changing his/her vote he/she can change the outcome of a vote. One counts the number of times a player is a swinger and divides it by the total number of swings for all of the players. This number is the player's Banzhaf number.

Consider the following example:

Suppose we are given a weighted majority game in which player 1 is given four votes, player 2 is given 3 votes, player 3 is given 2 votes, and player 4 is given 1 vote. Suppose we require a simple majority to win. In the notation previously introduced one can express this game as

$$[6; 4, 3, 2, 1]$$

In Table 1, we list all $2^4 = 16$ possible voting outcomes.

(Table 1 appears on the following page.)

Player 1 2 3 4	Issue Passed	Swinger 1	 2	 3	 4
N N N N	No				
N N N Y	No				
N N Y N	No	X			
N N Y Y	No	X	X		
N Y N N	No	X			
N Y N Y	No	X		X	
N Y Y N	No	X			X
N Y Y Y	Yes		X	X	X
Y N N N	No		X	X	
Y N N Y	No		X	X	
Y N Y N	Yes	X		X	
Y N Y Y	Yes	X		X	
Y Y N N	Yes	X	X		
Y Y N Y	Yes	X	X		
Y Y Y N	Yes	X			
Y Y Y Y	Yes				
	Total	10	6	6	2

Table 1

Thus the Banzhaf value for player 1, β_1, is

$$\beta_1 = \frac{\text{\# of times player 1 is a swinger}}{\text{total \# of times any player is a swinger}} = \frac{5}{12}$$

and similarly, $\beta_2 = \frac{1}{4}$, $\beta_3 = \frac{1}{4}$, $\beta_4 = \frac{1}{12}$

One can see from the definition that the β_i are nonnegative and

$$\sum_{i \in N} \beta_i = 1.$$

The Banzhaf value is symmetric (that is, if $w_i = w_j$ then $\beta_i = \beta_j$) and monotone (if $w_i > w_j$ then $\beta_i \geq \beta_j$). The Banzhaf index assigns a value of 1 to dictators[2] and 0 to dummies.

There are several assumptions inherent in our weighted voting scheme. First we assume that all the votes that a single player is given must be cast as a bloc. And a player must vote either yes or no. Abstentions and absenteeisms are not allowed. Also it is assumed that the legislator acts as a true delegate and votes according to the majority opinion in his district. A probabilistic interpretation can be made of the Banzhaf value. If one assumes that a voter will support or oppose an issue with probability $\frac{1}{2}$ (i.e. each outcome is equally likely), then the

$$\frac{\text{(number of times a voter is a swinger)}}{\text{(total number of outcomes)}}$$

is the probability a voter is a swinger.

Some concern has been expressed as to whether weighted voting systems are just in regard to a representative's voice in the legislature. Should legislator A be given 9 times as much time to debate on the issues as legislator B if district A is 9 times larger than district B? The New York Court of Appeals answered that question in the negative in Iannucci

---

[2] This assumes $\dfrac{\sum_{i=1}^{n} w_i}{2} < q$.

v. Board of Supervisors (1967) [7].

"Of course, in any weighted voting scheme, those representatives who cast the larger aggregate of votes can be expected to have greater influence with their colleagues than respresentatives with only a single vote. We find nothing unconstitutional in a disparity of influence among the various members of a county board of supervisors. In every legislature there will be some members, who because of particular expertise, wealth, political office, a reputation for probity and the like will be found to exercise more sway than others in the passage or defeat of legislation, and, when weighted voting is employed, such influence might well attach to the repesentatives from larger constituents who cast the larger aggregates of votes."

Another consideration (which actually applies in any apportionment scheme) is the division of districts. It is the opinion of the New York courts that to atomize sections of the population into too many districts reduces the effectiveness of local legislative bodies and participation by its members (Iaunnucci v. Board of Supervisors) [7]. They are also concerned that dividing districts into equal size may cause smaller towns to lose their identities if they were combined with larger industrial communities, thereby creating districts lacking in mutual sentiments or interests (Iaunnucci). Weighted voting systems can be designed to overcome these difficulties. One can create a reasonable number of districts (perhaps of unequal size to prevent aggregation of unlike interests), and give each member a number of votes so that his/her voting power is in accord with the population of his/her district.

It is a common belief that representation for districts of unequal size should vary as the reciprocal of the population. This is not a valid assumption if one uses the Banzhaf index to designate equality among voters.

Banzhaf [3] has illustrated how one's influence varies with population. His example is shown in Table 2.

District Symbol	No. of Voters	No. of Combin.	No. of Combin. that are swings	% of Combin. that are swings	Individ. % of Pop.	
	n	2^n	b	$100b/2^n$	$100/n$	$100/\sqrt{n}$
A	3	8	4	50.0	33.3	57.7
B	5	32	12	37.5	20.0	44.7
C	7	128	40	31.2	14.3	37.7
D	9	512	140	27.4	11.1	33.3

Table 2

These districts have only a few citizens and we are considering an issue with only two alternatives. (To construct an example with large populations is computationally infeasible.) In the table,

$$b = \frac{2(n-1)!}{\left(\frac{n-1}{2}\right)!\left(\frac{n-1}{2}\right)!}$$

In order for a player to be a swinger the rest of the players, (n-1 of them) must split half for the issue and half against. The factor of 2 in the numerator is due to the fact that a given group of voters can be in favor or against the issue.

In District A, there are 3 voters. Each voter has one vote and simple majority is required to win. There are $2^3 = 8$ number of outcomes. An individual voter is a swinger in four outcomes. Thus the percentage of time he/she is a swinger if $4/8 = 1/2$ (number of times a swinger/number of outcomes.) Yet the individual is 33.3 percent of the population. Thus we can see the percentage of time a voter is a swinger is closer to $100/\sqrt{n} = 57.7$ than to $100/n = 33.3$. And analyzing the other districts

similarly we see that in all cases the percentage of times a voter is a swinger is more closely alligned to $1/\sqrt{\text{population}}$ than it is to 1/population.[3] Thus if District 1 had four times the population as District 2, this suggests that District 1 should only have twice as many representatives as District 2.

One can also verify the square root effect by using Stirling's approximation

$$n! \overset{4}{\sim} e^{-n}\, n^{n}\quad 2\sqrt{\pi n}\quad \text{(consult any advanced calculus text).}$$

Suppose we have $n+1$ voters where n is even. Then a player i is a swinger if and only if the other players divide exactly half for the issue and half against the issue. Thus the number of swings for players i is

$$\frac{2(n!)}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}$$

where the factor of two is again due to the fact that either group can be for or against the issue.

Using Stirling's approximation for $n!$, we get

$$\frac{2(n!)}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!} \times \frac{1}{2^{n}} = (\#\text{ swingers for player } i)(1/\#\text{ of outcomes}) \approx \frac{2}{\sqrt{2\pi n}}$$

---

[3] We have assumed that the quota is a simple majority and each outcome is equally likely.

[4] $\sim$ means approximately equal to.

The

$$\frac{\text{(number of times a voter is a swinger)}}{\text{(total number of outcomes)}}$$

is the probability that a player is a swinger. The courts have mandated that the probability that a voter affects the outcome of a vote (In our terminology this is the probability of being a swinger.) be equal for all voters. The probability that a voter I will be a swinger in his/her District J is

$$\frac{\text{(number of times voter I is a swinger)}}{(2^{\text{(population of District J)}})} \; .$$

The probability that the legislator for District J is a swinger is

$$\frac{\text{(number of times legislator J is a swinger)}}{(2^{\text{(number of legislators)}})} \; .$$

Voter I plays a simple majority game where all players have one vote. Legislator J plays a weighted majority game. The two games are independent. Thus the probability that voter I of District J affects the outcome of the vote in the legislature is the product of those two numbers. Thus we want to make that product a constant for each voter. We want to find a set of weights for the legislators such that

$$\text{(1)} \quad \left(\frac{\text{number of swings for voter I}}{2^{\text{(population of District J)}}}\right)\left(\frac{\text{number of swings for legislator J}}{2^{\text{(number of legislators)}}}\right) = C.$$

where C is a constant. Since $2^{\text{(number of legislators)}}$ is a term common for all voters we can include it in the constant C. Thus we equivalently write (1) as

$$(2)\quad \left(\frac{\text{number of swings for voter I}}{2^{(\text{population of District J})}}\right)\left(\text{number of swings for legislator J}\right) = C$$

We have previously shown that the term

$$(3)\quad \left(\frac{\text{number of swings for voter I}}{2^{(\text{population of District J})}}\right) \approx \frac{1}{\sqrt{\text{population of District J}}}$$

Thus combining (2) and (3), we have

$$(4)\quad \frac{(\text{number of swings for legislator J})}{\sqrt{\text{population of District J}}} \approx C$$

Thus when (4) is approximately equal to C we have made each voter's ability to affect the outcome of a vote approximately equal.

ANALYSIS OF FOUR NEW YORK COUNTIES

The nucleus of many county governments in the state of New York has been a board of supervisors, made up of a supervisor from each town in the county and one or more supervisors from each city, if any. The application of the "one person, one vote" concept to local governments invalidated the apportionment plans of many county governments. A large proportion (approximately 1/3) of these counties chose to implement weighted voting systems. Four of those counties are analyzed below. All of the proposed systems were devised so that power varies as $1/\sqrt{\text{population}}$.

Saratoga County

In 1968, Saratoga County established a weighted voting system for the County Board of Supervisors (Local Law 2 of 1968). In 1971, the law was updated to be in accordance with the 1970 Federal Census. Local Law 2 of 1971 (which is presently in effect) mandates that there shall be a 23 member board. The system is as follows:

1) Each Municipality shall elect one Supervisor whenever its population is less than 12,000 according to the latest decennial census and such Supervisors shall cast one vote for each person in his/her district according to such census.

2) When any such Municipality shall have a population of 12,000 or more, one additional Supervisor shall be elected therefrom and one additional thereafter as each whole multiple of 12,000 is attained. Such Supervisor shall each cast that number of votes arrived at by dividing the total population of said municipality by the number of its Supervisor.

This system illustrates a misconception regarding the relationship

PRESENT SYSTEM FOR SARATOGA COUNTY

DISTRICT		POPULATION	SWINGS	B(I)	SWINGS(I)/SQRT(POP)	WEIGHT(I)
1	Saratoga Springs	18845	502770	0.16950	3662.45	188
2	Clifton Park	14667	366058	0.12341	3022.59	147
3	Milton	10450	254166	0.08569	2486.33	105
4	Moreau	10080	243890	0.08222	2429.20	101
5	Halfmoon	9287	223642	0.07540	2320.68	93
6	Waterford	7559	181346	0.06114	2085.82	76
7	Ballston	6729	159302	0.05371	1941.99	67
8	Mechanicville	6247	147146	0.04961	1861.71	62
9	Corinth	5783	137462	0.04634	1807.62	58
10	Stillwater	5023	118310	0.03989	1669.32	50
11	Greenfield	4542	106298	0.03584	1577.25	45
12	Saratoga	4208	99242	0.03346	1529.88	42
13	Malta	3813	89686	0.03024	1452.42	38
14	Charlton	3772	89686	0.03024	1460.29	38
15	Wilton	2975	70726	0.02384	1296.69	30
16	Galway	2506	58942	0.01987	1177.43	25
17	Northumberland	1779	39970	0.01347	947.65	17
18	Hadley	1128	25750	0.00868	766.70	11
19	Edinburg	844	21138	0.00713	727.60	9
20	Providence	639	16370	0.00552	647.59	7
21	Day	615	14338	0.00483	578.16	6

THE QUOTA FOR THE GAME IS 608

TABLE 3

PROPOSED SYSTEM FOR SARATOGA COUNTY

DISTRICT		POPULATION	SWINGS	B(I)	SWINGS(I)/SQRT(POP)	WEIGHT(I)
1	Saratoga Springs	18845	334735	0.09462	2438.39	54
2	Clifton Park	14667	293581	0.08298	2424.14	48
3	Milton	10450	247697	0.07002	2423.05	41
4	Moreau	10080	241273	0.06820	2403.14	40
5	Halfmoon	9287	234901	0.06640	2437.51	39
6	Waterford	7559	209693	0.05927	2411.86	35
7	Ballston	6729	197215	0.05575	2404.17	33
8	Mechanicville	6247	191037	0.05400	2417.03	32
9	Corinth	5783	184857	0.05225	2430.86	31
10	Stillwater	5023	172583	0.04878	2435.10	29
11	Greenfield	4542	160369	0.04533	2379.56	27
12	Saratoga	4208	154291	0.04361	2378.50	26
13	Malta	3813	148231	0.04190	2400.52	25
14	Charlton	3772	148231	0.04190	2413.53	25
15	Wilton	2975	130121	0.03678	2385.63	22
16	Galway	2506	118145	0.03340	2360.07	20
17	Northumberland	1779	100235	0.02833	2376.46	17
18	Hadley	1128	82415	0.02330	2453.87	14
19	Edinburg	844	70593	0.01995	2429.91	12
20	Providence	639	58783	0.01662	2325.42	10
21	Day	615	58783	0.01662	2370.36	10

THE QUOTA FOR THE GAME IS 296

TABLE 4

between voting power and the number of votes a district should cast. The actual game was not analyzed (since the computation time of the program is related to the quota), but an analagous game was run which scaled the weights down by a factor of 100 (and rounded to the nearest integer). The results are summarized in Table 3. As one can see Saratoga Springs (District 1) has six times as many swings/$\sqrt{\text{person}}$ than does Day. The percentage difference between the most powerful and the least powerful district is 4.6.

A proposed system based on the "square root factor" is shown in Table 4. Note that the percentage difference between the largest and the smallest district is only 4.6%. And the district with the greatest power only deviates from the mean power by 1.4%. The smallest district only deviates from the mean power by 3.3%.

Schoharie County

Schoharie County reapportioned its Board of Supervisors in February 1975 to comply with the mandate of the Supreme Court of the Third Judicial District of the State of New York and the constitutionality requirements established by the Supreme Court of the United States.

In matters requiring a simple majority they implemented the following plan:

(the above Table is found on the following page)

TOWN	POPULATION (1970) Federal Census	SUPERVISOR'S VOTE
Cobleskill	4573	523
Schoharie	3088	400
Middleburgh	2486	323
Richmondville	1903	251
Esperance	1567	208
Sharon	1566	208
Seward	1271	169
Wright	1086	145
Fulton	1060	141
Carlisle	1040	139
Gilboa	854	114
Jefferson	840	112
Summit	690	92
Broome	551	74
Conesville	489	65
Blenheim	260	35
		2999

A proposed system for Schoharie County is listed in Table 5. The percentage difference between the district with the largest and smallest power is 3.2%. The district with the largest power deviates from the mean by 1.9%, the smallest deviates 2.2% from the mean.

Fulton County

Up until 1969, the Fulton Board of Supervisors was comprised of 20 members each representing a town or city within the county (10 towns plus 10 wards within two cities). The Board, during November of 1968, received a summons and complaint filed by a local taxpayer charging unequal representation by the Board members.

Upon receipt the Board immediately proposed a weighted voting system. The plan was approved in February 1969 and put into effect in March of 1969. Since that time the plan has been updated twice to conform with current populations. An analysis of the most swings per square root person

varies as much as 400% between Johnstown and Bleeker. A proposed system is listed in Table 7. The disparity between the largest and smallest district is 4.9%. The district with the largest power deviates from the mean by 2.2%, the smallest deviates by 2.8% from the mean.

Cortland County

In 1971, a suit was brought against the Board of Supervisors by Mr. Slater. The court ruled that the one town-one vote voting scheme was unconstitutional. An interim plan was then set up which used proportional weights. The new apportionment plan which was finally devised (which involved some redistricting and used the 1970 Federal census) was implemented in January 1974. The new plan for issues requiring a simple majority is listed in Table 8. The system was devised to yield power proportional to 1/population. However since the populations are very close the disparity of swings per square root person is not nearly as dramatic as in the other cases. A proposed system for Cortland County is listed in Table 9. The percentage difference between the largest and smallest district is 3.2%. The deviation from the mean of the largest district is 1.4% and the smallest district is 1.9%.

PROPOSED PLAN FOR SCHOHARIE COUNTY

DISTRICT		POPULATION	SWINGS	B(I)	SWINGS(I)/SQRT(POP)	WEIGHT (I)
1	Cobleskill	4573	11773	0.11881	174.10	100
2	Schoharie	3088	9785	0.09875	176.08	85
3	Middleburgh	2486	8607	0.08686	172.62	76
4	Richmondville	1903	7439	0.07507	170.53	67
5	Esperance	1567	6805	0.06867	171.91	61
6	Sharon	1566	6805	0.06867	171.96	61
7	Seward	1271	6087	0.06143	170.74	55
8	Wright	1086	5645	0.05697	171.30	51
9	Fulton	1060	5565	0.05616	170.93	50
10	Carlisle	1040	5565	0.05616	172.56	50
11	Gilboa	854	4997	0.05043	170.99	45
12	Jefferson	840	4997	0.05043	172.41	45
13	Summit	690	4503	0.04544	171.43	41
14	Broome	551	3945	0.03981	168.06	36
15	Conesville	489	3811	0.03846	172.34	34
16	Blenheim	260	2761	0.02786	171.23	25

THE QUOTA FOR THE GAME IS 442

TABLE 5

PRESENT SCHEME FOR FULTON COUNTY

DISTRICT		POPULATION	SWINGS	B(I)	SWINGS(I)/SQRT(POP)	WEIGHT(I)
1	Johnstown	5750	188006	0.11491	2479.35	58
2	Gloversville 1	5181	165726	0.10129	2302.42	52
3	Mayfield	4522	141386	0.08642	2102.52	45
4	Gloversville 5	3877	121202	0.07408	1946.53	39
5	Broadalbin	3542	108066	0.06605	1815.79	35
6	Gloversville 6	3366	104918	0.06413	1808.39	34
7	Gloversville 3	3289	101742	0.06219	1774.06	33
8	Johnstown 3	3178	98478	0.06019	1746.88	32
9	Johnstown 2	3066	95370	0.05829	1722.37	31
10	Perth	2383	73342	0.04483	1502.42	24
11	Northampton	2379	73342	0.04483	1503.68	24
12	Gloversville 2	2039	60914	0.03723	1348.99	20
13	Johnstown 4	2033	60914	0.03723	1350.98	20
14	Gloversville 4	1925	57814	0.03534	1317.70	19
15	Johnstown 1	1768	54742	0.03346	1301.91	18
16	Oppenheim	1431	42458	0.02595	1122.38	14
17	Ephratah	1297	39406	0.02409	1094.19	13
18	Caroga	822	24214	0.01480	844.56	8
19	Stratford	495	15174	0.00927	682.02	5
20	Bleeker	294	8898	0.00544	518.94	3

THE QUOTA FOR THE GAME IS 264

TABLE 6

PROPOSED PLAN FOR FULTON COUNTY

	DISTRICT	POPULATION	SWINGS	B(I)	SWINGS(I)/SQRT(POP)	WEIGHT(I)
1	Johnstown	5750	140325	0.07832	1850.55	88
2	Gloversville 1	5181	133439	0.07447	1853.86	84
3	Mayfield	4522	123287	0.06881	1833.38	78
4	Gloversville 5	3877	114969	0.06416	1846.43	73
5	Broadalbin	3542	108347	0.06047	1820.51	69
6	Gloversville 6	3366	106701	0.05955	1839.13	68
7	Gloversville 3	3289	105045	0.05863	1831.65	67
8	Johnstown 3	3178	105045	0.05863	1863.37	67
9	Johnstown 2	3066	101837	0.05684	1839.16	65
10	Perth	2383	88859	0.04959	1820.29	57
11	Northampton	2379	88859	0.04959	1821.81	57
12	Gloversville 2	2039	82437	0.04601	1825.63	53
13	Johnstown 4	2033	82437	0.04601	1828.33	53
14	Gloversville 4	1925	79213	0.04421	1805.43	51
15	Johnstown 1	1768	76153	0.04250	1811.11	49
16	Oppenheim	1431	68219	0.03807	1803.37	44
17	Ephratah	1297	65029	0.03629	1805.66	42
18	Caroga	822	50811	0.02836	1772.24	33
19	Stratford	495	40051	0.02235	1800.16	26
20	Bleeker	294	30737	0.01715	1792.62	20

THE QUOTA FOR THE GAME IS 573

TABLE 7

PRESENT SYSTEM FOR CORTLAND COUNTY

DISTRICT	POPULATION	SWINGS	B(I)	SWINGS(I)/SQRT(POP)	WEIGHT(I)
3	2716	53817	0.05859	1032.65	31
7	2560	51547	0.05612	1018.79	27
13	2517	50357	0.05482	1003.73	26
14	2500	50357	0.05482	1007.14	26
2	2478	49221	0.05358	988.78	25
8	2467	49221	0.05358	990.98	25
18	2460	49221	0.05358	992.39	25
15	2444	49221	0.05358	995.63	25
16	2442	49221	0.05358	996.04	25
6	2442	49221	0.05358	996.04	25
5	2440	49221	0.05358	996.45	25
19	2434	48115	0.05238	975.26	24
12	2422	48115	0.05238	977.67	24
1	2406	48115	0.05238	980.92	24
17	2402	48115	0.05238	981.73	24
4	2284	45159	0.04916	944.92	22
9	2187	43493	0.04735	930.03	20
10	2153	43423	0.04727	935.83	19
11	2140	43423	0.04727	938.67	18

THE QUOTA FOR THE GAME IS 231

TABLE 8

PROPOSED PLAN FOR CORTLAND COUNTY

DISTRICT	POPULATION	SWINGS	B(I)	SWINGS(I)/SQRT(POP)	WEIGHT(I)
3	2716	51048	0.05541	979.52	32
7	2560	50676	0.05500	1001.57	27
13	2517	49724	0.05397	991.12	26
14	2500	49724	0.05397	994.48	26
2	2478	48968	0.05315	983.70	25
8	2467	48968	0.05315	985.89	25
18	2460	48968	0.05315	987.29	25
15	2444	48968	0.05315	990.52	25
16	2442	48968	0.05315	990.92	25
6	2442	48968	0.05315	990.92	25
5	2440	48968	0.05315	991.33	25
19	2434	48124	0.05223	975.44	24
12	2422	48124	0.05223	977.86	24
1	2406	48124	0.05223	981.10	24
17	2402	48124	0.05223	981.92	24
4	2284	46332	0.05029	969.47	22
9	2187	46192	0.05014	987.74	21
10	2153	46192	0.05014	995.51	20
11	2140	46192	0.05014	998.53	20

THE QUOTA FOR THE GAME IS 233

TABLE 9

CONCLUSIONS

Courts are the bodies which ultimately accept or reject a proposed voting system. In the past the courts have been mainly concerned with the "one person, one vote" concept. It is their opinion that other inequities that arise because of a representative's seniority or personal power are not unconstitutional. The courts have ruled that each person's voting power (that is, their likelihood of casting the deciding vote in an outcome.) should be equal. Since the Banzhaf index measures precisely this characteristic it seems ideal. Since we have shown that the Banzhaf index should vary as $1/\sqrt{\text{population}}$, we defined "equality" of voting power as each person's swings / $\sqrt{\text{population}}$, being equal. Even though a voting system should achieve this objective, there is no guarantee that a court will accept the system. They have ruled that "mathematical exactness is not a workable constitutional requirement." [13] Thus each case is judged individually--and it is impossible to infer that disparities of 10% are acceptable on grounds that it has been accepted previously.

One concern of the New York Courts has been the theoretical capability of a minority of people to be able to pass an issue. Unfortunately this seems to happen frequently when the Banzhaf index is made proportional to $1/\sqrt{\text{population}}$. In the Schoharie and Fulton systems approximately 42% of the population can pass an issue. In Saratoga county, 36% of the population can pass an issue. In the Cortland system 49% of the population can control the vote. (The Cortland County system is unique since the variation in population is very small.) This phenomena seems to be an inherent paradox of the Banzhaf index, as well as for such representative forms of government.

It remains an open question as to whether for a given set of districts with specified populations, one can always devise a system for which the swings/$\sqrt{\text{population}}$ are equal (within reasonable bounds) for all districts.

APPENDIX

Computing the Banzhaf Value

The Banzhaf value as described previously can be obtained from a simplistic calculation. However that procedure is a very inefficient calculation. First of all, the number of times a player can join a losing coalition and make it into a winning one is exactly equal to the number of times he/she is in a winning coalition that would lose if he/she left the coalition.

Consider the configurations in which player 2 is a swinger. [Refer to Table 1.]

N N Y Y	Y N N N	Y N N Y
N Y Y Y	Y Y N N	Y Y N Y

One can clearly see if player 2 is removed from each pair of outcomes the configurations which are left in each pair are the same. In each case, one of the outcomes was winning and one was losing. Thus one only needs to consider either all the losing coalitions or all the winning coalitions. Since the player's Banzhaf value is the number of times he/she is a swinger/the total number of swings, we have reduced the numerator and denominator by a factor of two, so the Banzhaf index has not changed.

Now we turn to the actual counting of the swings. As a consequence of work done by David G. Cantor and Mann and Shapely for the Shapley value [10,11] we can obtain an efficient way to compute the Banzhaf value.

Let

$C^{i}_{j,s}$ = number of ways in which s players other than i, can have a sum of weights equal to j.

Then $SWINGS(I) = \sum_{s=0}^{n-1} \sum_{j=q-w_i}^{q-1} C^i_{j,s}$ = number of swings for player i. The inside summation counts the number of losing coalitions that have enough weight so that if the weight of player i is added to that coalition, it will reach or exceed the quota. (Thus it becomes a winning coalition.) We sum that quantity over all coalitions of size 0 to n-1. Thus

$$\beta(I) = \frac{SWINGS(I)}{\sum_{i\epsilon I} SWINGS(i)}$$

Cantor's contribution was to show the $C^i_{j,s}$ can be obtained from the generating function

$$f_i(x,y) = \prod_{\substack{k\epsilon N\\ k\neq i}} (1+x^{w_k}y)$$

where the product is taken over $k \epsilon N - \{i\}$ and w_k is the population of kth district. For any $f_i(x,y)$ can be obtained from the n-fold product.

$$\prod_{k\epsilon N} (1+x^{w_k}y)$$

divided by

$$(1+x^{w_i}y).$$

The $C^i_{j,s}$ can be found as elements of a matrix C^i of integers. For each player i, this matrix can be generated inductively as follows. Define $C^{(0)}$ so that $C^{(0)}_{0,0} = 1$ and all other $C^{(0)}_{j,s} = 0$. Then $C^{(r)}$ is obtained

from $C^{(r-1)}$ by the relation

$$C^{(r)}_{j,s} = C^{(r-1)}_{j,s} + C^{(r-1)}_{j-w_p\ ,\ s-1} \tag{1}$$

where the last term is taken to be 0 when either of its subscripts is negative and the w_p stands for the weights of the distinct players in $N - \{i\}$. $C^{(n-1)} = C^i$. Thus one can generate $C^{(n)}$ by taking all $r \in N$, and then obtain each C^i by subtracting once, by "reversing" the recursive relation (1).

The above matrix was developed for the Shapley value, which requires that one know the size of a particular coalition in which a player is a swinger. However this information is not required for the Banzhaf value. Thus we may collapse the matrix into a vector saving computational time. Thus recursion (1) becomes

$$\tilde{C}^{(r)}_{j} = \tilde{C}^{(r-1)}_{j} + \tilde{C}^{(r-1)}_{j-w_p} \tag{2}$$

where the last term is taken to be 0 if $j-w_p < 0$. After one has generated $\tilde{C}^{(n)}$ by taking all $r \in N$, $\tilde{C}^i$ may be obtained by "reversing" the recursive relation (2). Then

$$\text{SWINGS(I)} = \sum_{j=q-w_i}^{q-1} \tilde{C}^i_j$$

where w_i is the weight of player i and q is the quota for the game.

Another computational aid may be employed to reduce the number of calculations. It is best explained by example. Once again consider the game

[6; 4, 3, 2, 1]

Generating $\tilde{C}^{(4)}$ we obtain

$$\tilde{C}^{(0)} = [1,0,0,0,0,0]$$
$$\tilde{C}^{(1)} = [1,0,0,1,0,0] \qquad (\text{for } w_1 = 4)$$
$$\tilde{C}^{(2)} = [1,0,1,1,0,1] \qquad (\text{for } w_2 = 3)$$
$$\tilde{C}^{(3)} = [1,1,1,2,1,1] \qquad (\text{for } w_3 = 2)$$
$$\tilde{C}^{(4)} = [1,1,1,2,2,2] \qquad (\text{for } w_4 = 1)$$

To compute SWINGS(1) from $\tilde{C}^{(4)}$ we could reverse the recursion (2) to obtain

$$\tilde{C}^{1} = [1,1,1,2,1,1]$$

However there is another method which makes "reversing" the recursion unnecessary. Define a function F(i) where

$$F(i) = \sum_{j=0}^{i} \tilde{C}_j^{(n)}$$

Thus for our example,

$$F(0) = 1 \quad F(2) = 3 \quad F(4) = 7$$
$$F(1) = 2 \quad F(3) = 5 \quad F(5) = 9$$

Then $\text{SWINGS}(I) = F(Q-1) + \sum_{j=1}^{[\frac{Q-1}{w_i}]} 2(-1)^j F(Q-1-j(w_i)).$

Thus SWINGS(1) = 9-4 = 5

(2) = 9-6 = 3

(3) = 9-10+4 = 3

(4) = 9-14+10-6+4-2 = 1

To see why this works, look at the generating function for this games

$$\prod_{i=1}^{4} (1+x^{w_i})$$

(Since we do not need to keep track of the size of the coalitions, we do not need the y term.) For the game above we have

$$(1+x)(1+x^2)(1+x^3)(1+x^4) = 1 + x + x^2 + 2x^3 + 2x^4 + 2x^5 + 2x^6 + 2x^7 + x^8 + x^9 + x^{10} \qquad (3)$$

Dividing (3) by $(1+x^4)$ to obtain $\tilde{C}^1$, we get

$$1 + x + x^2 + 2x^3 + x^4 + x^5 + x^6$$

Looking at this more generally, suppose

$$\tilde{C}^{(n)} = c^0 + c_1x + c_2x^2 + c_3x^3 + \ldots + c_{10}x^{10}$$

Then dividing $\tilde{C}^{(n)}$ by $(1+x^4)$ we obtain

$$c_0 + c_1 + x + c_2x^2 + c_3x^3 + (c_4-c_0)x^4 + (c_5-c_1)x^5 + \ldots$$

(We only need the first six terms of the quotient for our later calculations.)

$$\begin{aligned} \text{Thus SWINGS}(1) &= c_2 + c_3 + (c_4-c_0) + (c_5-c_1) \\ &= c_5 + c_4 + c_3 + c_2 - c_1 - c_0 \\ &= F(5)-F(4)+F(4)-F(3)+F(3)-F(2)+F(2)-F(1)-F(1)+F(0)-F(0) \\ &= F(5)-2F(1) \end{aligned}$$

To compute $\tilde{C}^3$, we have

$$\tilde{C}^{(n)}/(1+x^2) = c_0+c_1+(c_2-c_0)x^2+(c_3-c_1)x^3+(c_4-c_2+c_0)x^4+(c_5-c_3+c_1)x^5+\ldots$$

$$\begin{aligned} \text{Therefore SWINGS}(3) &= c_5+c_4-c_3-c_2+c_1+c_0 \\ &= F(5) - 2F(3) + 2F(1) \end{aligned}$$

To prove this generally, the notation becomes cumbersome, so we leave it to the reader to verify.

Finding an initial solution

In a game with a large number of players constructing an initial set of weights for a voting system may be tricky. For instance in the Saratoga game changing player 1's weight by one vote changes the number of swings for each player as follows.

District	SWINGS WHEN $w_1 = 55$	SWINGS WHEN $w_1 = 54$
1	340,602	334,735
2	292,562	293,581
3	246,906	247,697
4	240,486	241,273
5	234,170	234,901
6	209,054	209,693
7	196,594	197,215
8	190,406	191,037
9	184,310	184,857
10	172,046	172,583
11	159,866	160,369
12	153,806	154,291
13	147,782	148,231
14	147,782	148,231
15	129,714	130,121
16	117,770	118,145
17	99,934	100,235
18	82,162	82,415
19	70,382	70,593
20	58,602	58,793
21	58,602	58,793

Systems with large numbers of players and fairly large quotas seem very volatile. Thus it is impossible to analyze the impact of a shift in weights without a computer analysis. This flucuation compounds the problem of finding an initial solution.

The following heuristic worked well in the analysis of the foremen-tioned counties. We have already shown that voting power and voting weight are not synonymous. However if the weights are nearly equal, the correspondence remains close. (Of course a set of weights could be so close that though there is a variation in weights there is no variation in power.)

Arbitrarily choose a weight for the player with the smallest population. (One wants to compromise between keeping the quota small and avoidng round-off error when choosing this weight.) Call this player, s, and his/her weight, w_s. Then assign player i the following weight

$$w_s \left(\frac{\sqrt{p_i}}{\sqrt{p_s}} \right)$$

where p_i is the population of district i. One should check to be sure that the weights will not give every player the same power. (In the Cortland game, the above heuristic generated a game in which any ten players could form a winning coalition.) After the initial weights are assigned, a computer analysis must be made. Then it is a trial and error process until an acceptable set of weights are found.

Computer Documentation

Both programs which appear in this appendix have been written in PL/C. They are also compatible with PL/1. For either of the programs the data should be submitted in the following format: number of players (districts), quota for the game, weights for the players(districts) in monotonically decreasing order, and respective populations of the players (districts).

The first program does not take advantage of the computational aids for the Banzhaf value (presented in section (1) in this appendix.) However it may be adapted to compute the Shapley value by adding a procedure which generates the proper binomial coefficients for a given coalition size. C(NEW,M,S) = $C^{(n)}_{m,s}$ after statement 36 in the program. C(OLD,M,S) = $C^{i}_{m,s}$ after statement 59.

The second program is much more efficient. On Cornell University's IBM 370, all programs ran in less than a second. The program employs both of the computational aids described earlier. After statement 32, C(OLD,J) = $\tilde{C}^{(n)}_{j}$.

The output for both programs is self-explanatory. Reading across line k of the output we have: k, the number of the district, its corresponding population, the number of swings for district k, the Banzhaf index, $B(k)$, for district k, (swings for district k)/ $\sqrt{(\text{population of district } k)}$, and the weight assigned to district k.

```
GFBZHAF: PROC OPTIONS(MAIN);
  DCL NUMBER             FIXED BIN,      /* # OF PLAYERS                   */
      QUOTA              FIXED BIN;      /*QUOTA OF THE GAME               */
  GET FILE(SYSIN) LIST(NUMBER,QUOTA);
  CALL GENFUNC;
GENFUNC: PROCEDURE;
  DCL WEIGHT(0:NUMBER) FIXED BIN,        /*WEIGHT OF PLAYER(I)             */
      SWINGS(0:NUMBER) FIXED BIN(31),    /* # OF SWINGS FOR PLAYER(I)      */
      C(0:1,0:QUOTA,0:NUMBER)
                         FIXED BIN       /*MATRIX USED TO COMPUTE SWINGS*/
                         INIT((2*(NUMBER+1)*(QUOTA+1))0) ,
      Q                  FIXED BIN       /* SUM OF THE WEIGHTS             */
                         INIT(0),
      B                  FLOAT BIN,      /*BANZHAF VALUE                   */
      TOTAL_SWINGS       FIXED BIN(31),  /*SUM OF SWINGS                   */
      M_WEIGHT           FIXED BIN,      /*TEMP VAR =M-WEIGHT(I)           */
      POP(NUMBER)        FIXED BIN(31);  /*POPULATION OF PLAYER(I)         */
  /* READ IN WEIGHTS */
    DO I=1 TO NUMBER;
       GET FILE(SYSIN) LIST(WEIGHT(I));
       Q=Q+WEIGHT(I);
       END;
  /*CHECK TO SEE IF QUOTA IS SIMPLE MAJORITY OF THE GIVEN WEIGHTS     */
    IF QUOTA¬=FLOOR(Q/2) + 1
       THEN DO;
          PUT FILE(SYSPRINT) PAGE LIST
                        ('QUOTA IS INCORRECT FOR GIVEN WEIGHTS');
          RETURN;
          END;
  /*READ IN POPULATIONS */
    DO I=1 TO NUMBER;
       GET FILE(SYSIN) LIST(POP(I));
       END;
  /*THE  # OF SWINGS IN COMPUTED BY USING GENERATING FUNCTIONS        */
  /*IT IS DONE  BY USING A MATRIX C    AND GENERATED RECURSIVELY      */
  /*AS FOLLOWS C(NEW,I,J)=C(OLD,I,J) + C(OLD,I,J-WEIGHT(I)) WHERE THE*/
  /* TERM IS TAKEN AS ZERO IF J-WEIGHT(I)  < 0                        */
    NEW=1;
    OLD=0;
    C(NEW,0,0)=1;
    C(OLD,0,0)=1;
    DO I=1 TO NUMBER;
       DO M= 0 TO QUOTA;
          DO S= 0 TO NUMBER;
             IF ((S-1>=0) & (M-WEIGHT(I)>=0))
                THEN  C(NEW,M,S)=C(OLD,M,S) + C(OLD,M-WEIGHT(I),S-1);
                ELSE C(NEW,M,S)=C(OLD,M,S);
             END;
          END;
       IF I< NUMBER THEN DO;
          OLD=NEW;
          NEW=1-OLD;
          END;
       END;
```

```
/*COMPUTE SWINGS FOR THE NUMBERTH PLAYER        */
   SWINGS(NUMBER)=0;
   DO M= QUOTA-WEIGHT(NUMBER) TO QUOTA -1;
      DO S= 0 TO NUMBER-1;
         SWINGS(NUMBER)=SWINGS(NUMBER) + C(OLD,M,S);
         END;
      END;
   TOTAL_SWINGS=SWINGS(NUMBER);
/*COMPUTE SWINGS FOR PLAYERS          */
   DO I=NUMBER-1 TO 1 BY -1;
      IF WEIGHT(I)= WEIGHT(I+1)
         THEN SWINGS(I)=SWINGS(I+1);
         ELSE DO;
            DO M=0 TO WEIGHT(I)-1;
               DO S= 0 TO NUMBER;
                  C(OLD,M,S)=C(NEW,M,S);
                  END;
               END;
            DO M=WEIGHT(I) TO QUOTA;
               M_WEIGHT=M-WEIGHT(I);
               DO S=1 TO NUMBER;
                  C(OLD,M,S)=C(NEW,M,S)-C(OLD,M_WEIGHT,S-1);
                  END;
               END;
            SWINGS(I)=0;
            DO M=(QUOTA-WEIGHT(I)) TO QUOTA-1;
               DO S=0 TO NUMBER;
                  SWINGS(I)=C(OLD,M,S)+ SWINGS(I);
                  END;
               END;
            TOTAL_SWINGS=TOTAL_SWINGS + SWINGS(I);
            END;
      END;
/* COMPUTE AND PRINT BANZHAF VALUES    */
   PUT FILE(SYSPRINT) PAGE EDIT('DISTRICT','POPULATION','SWINGS',
   'B(I)','SWINGS(I)/SQRT(POP)', 'WEIGHT(I)')(COL(12),A,X(6),A,X(6),
         A,X(14),A,X( 7),A,X(6),A);
   DO I= 1 TO NUMBER;
      B=SWINGS(I)/FLOAT(TOTAL_SWINGS);
      PUT FILE(SYSPRINT)  SKIP EDIT(I,POP(I),SWINGS(I), B,
            SWINGS(I)/FLOAT(SQRT(POP(I))),WEIGHT(I))
                  (COL(3),F(13),F(15),F(15),F(21,5),F(19,2),F(18));
      END;
   PUT FILE(SYSPRINT) EDIT('THE QUOTA FOR THE GAME IS',QUOTA)
                   (COL(12),A,F(6));
END GENFUNC;
END GFBZHAF;
```

```
GFBZHAF: PROC OPTIONS(MAIN);
  DCL NUMBER              FIXED BIN,      /* # OF PLAYERS                     */
      QUOTA               FIXED BIN,      /*QUOTA OF THE GAME                 */
      QUOTA_1             FIXED BIN;      /*NAME FOR QUOTA-1                  */
  GET FILE(SYSIN) LIST(NUMBER,QUOTA);
  QUOTA_1=QUOTA-1;
  CALL GENFUNC;
GENFUNC: PROCEDURE;
  DCL WEIGHT(0:NUMBER)  FIXED BIN,      /*WEIGHT OF PLAYER(I)              */
      SWINGS(0:NUMBER)  FIXED BIN(31),  /* # OF SWINGS FOR PLAYER(I)       */
      C(0:1,0:QUOTA)    FIXED BIN       /*MATRIX USED TO COMPUTE SWINGS*/
                        INIT((2*(QUOTA+1))0),
      Q                 FIXED BIN       /* SUM OF THE WEIGHTS              */
                        INIT(0),
      SIGN              FIXED BIN,      /* DETERMINES SIGN OF F(K)         */
      F(0:QUOTA)        FIXED BIN(31),  /*F(K)=C(OLD,0)+...C(OLD,K)        */
      WEIGHT_T          FIXED BIN,      /*TEMP VARIABLE FOR WEIGHT(I)      */
      WEIGHT_SUM        FIXED BIN,      /* TEMP VAR=QUOTA-1-WEIGHT(I)      */
      B                 FLOAT BIN,      /*BANZHAF VALUE                    */
      TOTAL_SWINGS      FIXED BIN(31),  /*SUM OF SWINGS                    */
      POP(NUMBER)       FIXED BIN(31);  /*POPULATION OF PLAYER(I)          */
  /* READ IN WEIGHTS */
    DO I=1 TO NUMBER;
      GET FILE(SYSIN) LIST(WEIGHT(I));
      Q=Q+WEIGHT(I);
      END;
  /*CHECK TO SEE IF QUOTA IS SIMPLE MAJORITY OF THE GIVEN WEIGHTS      */
    IF QUOTA¬=FLOOR(Q/2) + 1
      THEN DO;
        PUT FILE(SYSPRINT) SKIP(2) LIST
                           ('QUOTA IS INCORRECT FOR GIVEN WEIGHTS');
        RETURN;
        END;
  /*READ IN POPULATIONS */
    DO I=1 TO NUMBER;
      GET FILE(SYSIN) LIST(POP(I));
      END;
  /*THE  # OF SWINGS IN COMPUTED BY USING GENERATING FUNCTIONS          */
  /*IT IS DONE  BY USING A MATRIX C    AND GENERATED RECURSIVELY         */
  /*AS FOLLOWS C(NEW,J)=C(OLD,J) + C(OLD,J-WEIGHT(I)) WHERE THE LAST */
  /* TERM IS TAKEN AS ZERO IF J-WEIGHT(I)  < 0                          */
    NEW=1;
    OLD=0;
    C(NEW,0)=1;
    C(OLD,0)=1;
    DO I=1 TO NUMBER;
      WEIGHT_T=WEIGHT(I);
      DO J=0 TO QUOTA;
        IF (J-WEIGHT_T >=0)
          THEN C(NEW,J)=C(OLD,J)+ C(OLD,J-WEIGHT_T);
          ELSE C(NEW,J)=C(OLD,J);
        END;
      OLD=NEW;
      NEW=1-OLD;
      END;
```

```
/*LET F(N)= SUM OF C(OLD,0)+...+C(OLD,N)   */
   F(0)=C(OLD,0);
   DO I= 1 TO QUOTA;
      F(I)=F(I-1) + C(OLD,I);
      END;
/* USE F(*) TO COUNT SWINGS FOR THE PLAYERS                              */
   WEIGHT(0)=0;   /*DUMMY VARIABLE NEEDED TO MAKE COMPARISON             */
   TOTAL_SWINGS=0;
   DO I= 1 TO NUMBER;
      IF WEIGHT(I)¬=WEIGHT(I-1)
         THEN DO;
            SIGN=-2;            /*DETERMINES THE SIGN OF F(K)            */
            WEIGHT_T=WEIGHT(I);
            WEIGHT_SUM=QUOTA_1-WEIGHT_T;
            SWINGS(I)=F(QUOTA_1);
            DO WHILE(WEIGHT_SUM>=0);
               SWINGS(I)=SWINGS(I)+SIGN*F(WEIGHT_SUM);
               SIGN=-SIGN;
               WEIGHT_SUM=WEIGHT_SUM-WEIGHT_T;
               END;
            END;
         ELSE SWINGS(I)=SWINGS(I-1);
      TOTAL_SWINGS=TOTAL_SWINGS + SWINGS(I);
      END;
/*COMPUTE AND PRINT BANZHAF VALUES      */
   PUT FILE(SYSPRINT) PAGE EDIT('DISTRICT','POPULATION','SWINGS',
   'B(I)','SWINGS(I)/SQRT(POP)','WEIGHT(I)') (COL(12),A,X(6),A,X(6),A,
    X(14),A,X( 7),A,X(6),A);
   DO I= 1 TO NUMBER;
      B=SWINGS(I)/FLOAT(TOTAL_SWINGS);
      PUT FILE(SYSPRINT)  SKIP EDIT(I,POP(I),SWINGS(I), B,
                SWINGS(I)/FLOAT(SQRT(POP(I))),WEIGHT(I))
                   (COL(3),F(13),F(15),F(15),F(21,5),F(19,2),F(18));
      END;
   PUT FILE(SYSPRINT) SKIP(2) EDIT('THE QUOTA FOR THE GAME IS',QUOTA)
                      (COL(12),A,F(6));
END GENFUNC;
END GFBZHAF;
```

REFERENCES

[1] Baker v. Carr, 206 Federal Supplement 341 (1962).

[2] Banzhaf, John F., III, "Weighted Voting Doesn't Work: A Mathematical Analysis, Rutgers Law Review," Vol. 19, 1965, pages 317-343.

[3] Banzhaf, John F., III, "Multi-Member Electorial Districts--Do They Violate the "One Man--One Vote" Principle," The Yale Law Journal, Vol. 75, 1966, pages 1309-1338.

[4] Bell, Roderick, Edwards, David V., and Wagner, R. Harrison, editors, Political Power: A Reader in Theory and Research, The Free Press, New York, 1969.

[5] Graham v. Board of Supervisors of Erie County, 267 New York Supplement 2nd ed., 383 (1967).

[6] Gray v. Sanders, 372 United States Reports 368, 381 (1963).

[7] Iannucci v. Board of Supervisors, 282 New York Supplement 2nd ed. 502 (1967).

[8] Johnson, Ronald E., "An Analysis of Weighted Voting as Used in Reapportionment of County Governments in New York State," Albany Law Review, Vol. 34, No. 1, 1969, pages 1-45.

[9] Lucas, William F. "Measuring Power in Weighted Voting Systems," Technical Report No. 227, School of Operations Research and Industrial Engineering, Cornell University, September 1974.

[10] Mann, Irwin and Shapley, L.S., " Values of Large Games, IV." Evaluating the Electorial College by Morte Carlo Techniques," Rm 2651, The RAND Corporation, Santa Monica, California, Sept., 1960.

[11] Mann, Irwin and Shapley, L.S. "Values of Large Games, VI: Evaluating the Electorial College Exactly," Rm-3158-PR, The RAND Corporation, May, 1962.

[12] "Reapportionment: Local Government Legislative Bodies." Information Bulletin #23. Office for Local Government, Executive Department State of New York. Albany, New York.

[13] Reynold v. Sims, 377 Supreme Court Reporter 1362 (1964).

[14] Riker, William H. "Some Ambiguities in the Notion of Power," American Political Science Review, Vol. 58, 1964, pages 341-349.

[15] Riker, William H. and Shapley, Lloyd S. "Weighted Voting: A Mathematical Analysis for Instrumental Judgement" Chapter 15 in Representation Nomasx, edited by J. Roland Pennock and John W. Chapmen, Atherton Press, Inc., 1968: also as The RAND Corporation paper p-3318, Santa Monica, California, March, 1966.

[16] Slater v. Board of Supervisors of Cortland County, 346 New York Supplement 2 ed. 185 (1971).

[17] Town of Greenburgh v. Board of Supervisors, 293 New York Supplement 2nd ed. 957 (1968).

Unclassified (14) TR-309

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 309	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) (6) AN ANALYSIS OF WEIGHTED VOTING SYSTEMS USING THE BANZHAF VALUE.		5. TYPE OF REPORT & PERIOD COVERED (9) Technical Report,
		6. PERFORMING ORG. REPORT NUMBER Technical Report No. 309
7. AUTHOR(s) (10) Eleanor Ann Walther		8. CONTRACT OR GRANT NUMBER(s) (15) N00014-67-A-0077-0014, NSF-MPS-75-02024
9. PERFORMING ORGANIZATION NAME AND ADDRESS School of Operations Research and Ind. Engineering Cornell University Ithaca, NY 14853		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS (11) Sep 76
11. CONTROLLING OFFICE NAME AND ADDRESS Operations Research Program Office of Naval Research Arlington, VA 22217		12. REPORT DATE September 1976
		13. NUMBER OF PAGES (12) 48p.
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

game theory; simple games; weighted ~~voting; Banzhaf~~ value.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In 1962, the Supreme Court in its decision of Baker v. Carr opened the door for reapportionment by population. In 1967, the New York Courts ruled that weighted voting systems could be implemented in reapportionment schemes. The New York Courts have also accepted the Banzhaf value as a measure of → next page

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-014-6601

unclassified 403133
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

cont.

voting power.

This thesis defines the Banzhaf value and discusses some of its properties. An analysis of existing weighted voting schemes for four New York counties are presented. New schemes are proposed. The appendix presents the method used to compute the Banzhaf value. The computer programs are also included in the appendix.